

GROUPS WITH A NONTRIVIAL NONIDEAL KERNEL

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ABSTRACT. We classify finite groups G , such that the group algebra, $\mathbb{Q}G$ (over the field of rational numbers \mathbb{Q}), is the direct product of the group algebra $\mathbb{Q}[G/N]$ of a proper factor group G/N , and some division rings.

1. INTRODUCTION

Let G be a finite group and \mathbb{K} a field of characteristic zero. By Maschke's theorem and Wedderburn-Artin theory, the group algebra $\mathbb{K}G$ of G over \mathbb{K} is a direct product of matrix rings over division algebras:

$$\mathbb{K}G \cong \mathbf{M}_{d_1}(D_1) \times \cdots \times \mathbf{M}_{d_r}(D_r).$$

A natural question to ask is when each factor in this decomposition is actually a division ring (equivalently, the group algebra $\mathbb{K}G$ contains no nilpotent elements). In the classical case where \mathbb{K} is algebraically closed, it is well known that $\mathbb{K}G$ is a direct product of division rings if and only if G is abelian. For $\mathbb{K} = \mathbb{Q}$, the question was solved by S. K. Seghal [13, Theorem 3.5] (see Theorem 4.5 below).

In this paper, we consider a slightly more general question: Let $1 \neq N \trianglelefteq G$ be a normal subgroup. Then

$$\mathbb{K}G \cong \mathbb{K}[G/N] \times I,$$

where the (twosided) ideal I is the kernel of the canonical homomorphism $\mathbb{K}G \rightarrow \mathbb{K}[G/N]$. Now we ask: for which finite groups is there an $N \neq 1$ such that the ideal I above is a direct product of division rings? If there is such an N , then any nilpotent element of $\mathbb{K}G$ has constant coefficients on cosets of N . Also, only twosided ideals of $\mathbb{K}G$ can distinguish the elements of N .

The following is just a basic observation, which allows us to state our results more conveniently.

Lemma A. *For each field \mathbb{K} (of characteristic zero) and each finite group G , there is a unique maximal normal subgroup N , denoted by $\text{NKer}_{\mathbb{K}}(G)$, such that the kernel of the map $\mathbb{K}G \rightarrow \mathbb{K}[G/N]$ is a direct product of division rings.*

2010 *Mathematics Subject Classification.* 20C15.

Key words and phrases. Characters, Finite groups, Representations, Schur indices, Division rings.

Author supported by the DFG (Project: SCHU 1503/6-1).

We will give a more direct definition of $\text{NKer}_{\mathbb{K}}(G)$ in Section 3 below, before we prove Lemma A. We call $\text{NKer}_{\mathbb{K}}(G)$ the **nonideal kernel** of G (over \mathbb{K}).

We view the zero ideal as an empty product of division rings, so possibly $\text{NKer}_{\mathbb{K}}(G) = 1$. Indeed, this is the case for “most” groups, and we want to classify the groups G for which $\text{NKer}_{\mathbb{K}}(G) \neq 1$. Our first result concerns the field \mathbb{R} of real numbers.

We need to recall a definition: A nonabelian group G is called **generalized dicyclic**, if it has an abelian subgroup A of index 2 and an element $g \in G \setminus A$ such that $g^2 \neq 1$ and $a^g = a^{-1}$ for all $a \in A$. If A is cyclic, then G is called **dicyclic** (or generalized quaternion). Furthermore, Q_8 denotes the quaternion group of order 8 and C_n a cyclic group of order n .

Theorem B. *Let G be a finite group. Then $\text{NKer}_{\mathbb{R}}(G) > 1$ if and only if one of the following holds:*

- (i) G is abelian and $G \neq \{1\}$.
- (ii) G is generalized dicyclic.
- (iii) $G \cong C_4 \times Q_8 \times (C_2)^r$, $r \in \mathbb{N}$.
- (iv) $G \cong Q_8 \times Q_8 \times (C_2)^r$, $r \in \mathbb{N}$.

The motivation for this work is a question of Babai [1]. Babai asked which finite groups are isomorphic to the affine symmetry group of an orbit polytope. (An orbit polytope is a polytope such that its (affine) symmetry groups acts transitively on the vertices of the polytope.) In joint work with Erik Friese [6] (continuing our earlier paper [5]), we develop a general theory, which shows, among other things, that G is isomorphic to the affine symmetry group of an orbit polytope when $\text{NKer}_{\mathbb{R}}(G) = 1$. When $\text{NKer}_{\mathbb{R}}(G) > 1$, this may or may not be the case. Theorem B above is an essential ingredient in our answer to Babai’s question. Similarly, when $\text{NKer}_{\mathbb{Q}}(G) = 1$, then G can be realized as the affine symmetry group of an orbit polytope with vertices having rational coordinates.

The classification of groups with $\text{NKer}_{\mathbb{Q}}(G) > 1$ is more complicated. To state it, we first describe a special type of such groups.

Lemma C. *Let p and q be primes, let $P = \langle g \rangle \times P_0$ be an abelian p -group and Q an abelian q -group. Suppose P acts on Q such that $x^g = x^k$ for all $x \in Q$ and some integer k independent of $x \in Q$, and such that $\mathbf{C}_P(Q) = \langle g^{p^c} \rangle \times P_0$. Suppose that $p^d = \text{ord}(g^{p^c})$ is the exponent of $\mathbf{C}_P(Q)$, and that $(q-1)_p$, the p -part of $q-1$, divides p^d . Then for the semidirect product $G = PQ$, we have $1 \neq \text{NKer}_{\mathbb{Q}}(G) \cap \langle g \rangle$.*

Notice that the assumption on the action of g on Q and $|P/\mathbf{C}_P(Q)| = p^c$ imply that p^c divides $q-1$, and that the multiplicative order of k modulo the exponent of Q is just p^c . One can show that $\text{NKer}_{\mathbb{Q}}(G) = \langle g^{p^s} \rangle$, where $p^s = p^{c-1}(q-1)_p$. Whenever we mention “groups as in Lemma C”, we also use the notation established in the statement of Lemma C.

Theorem D. *Let G be a finite group. Then $\text{NKer}_{\mathbb{Q}}(G) \neq 1$ if and only if at least one of the following holds:*

- (i) G is abelian.
- (ii) $G = S \times A$, where S is a 2-group of exponent 4 which appears on the list from Theorem B, the group A is abelian of odd order, and the multiplicative order of 2 modulo $|A|$ is odd.
- (iii) G is generalized dicyclic.
- (iv) $G = (PQ) \times B$, where the subgroups $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$ and B are abelian, PQ is as in Lemma C, and the p -part of the multiplicative order of q modulo $|B|$ divides the multiplicative order of q modulo p^d .
- (v) $G = Q_8 \times (C_2)^r \times H$, where H is as in (iv) and has odd order, and the multiplicative order of 2 modulo $|H|$ is odd.

Case (ii) contains the groups $G = Q_8 \times (C_2)^r \times A$, for which $\mathbb{Q}G$ is a direct product of division rings, as classified by Sehgal [13].

An important tool in the proofs of Theorems B and D is Blackburn's classification of finite groups in which all nonnormal subgroups have a nontrivial intersection [3]. As we will see below, $\text{NKer}_{\mathbb{K}}(G)$ is always contained in the intersection of all nonnormal subgroups of G . While the proof of Theorem B is relatively elementary, the proof of Theorem D also depends on some deep facts about division algebras and Schur indices.

The paper is organized as follows: In Section 2, we review some basic facts about representations and characters over fields not necessarily algebraically closed, and in particular Schur indices. We also introduce the auxiliary concept of *skew-linear characters*. In Section 3, we define $\text{NKer}_{\mathbb{K}}(G)$ and prove some elementary properties. In Section 4, we consider Dedekind groups (groups such that all subgroups are normal). In such groups, we have either $\text{NKer}_{\mathbb{K}}(G) = 1$, or $\text{NKer}_{\mathbb{K}}(G) = G$, where the latter are exactly the groups such that $\mathbb{K}G$ is a direct product of division rings. Finally, Section 5 contains the proof of Theorem B, and Section 6 the (long) proof of Theorem D.

2. SKEW-LINEAR CHARACTERS

Let G be a finite group. For simplicity, assume that $\mathbb{K} \subseteq \mathbb{C}$ and write $\text{Irr } G$ for the set of irreducible complex characters of G . We begin by reviewing the relation between the representation theory of G over \mathbb{K} and over \mathbb{C} [8, § 38][9, Chapter 10].

By Maschke's theorem and general Wedderburn-Artin theory, the group algebra $\mathbb{K}G$ is the direct product of simple rings:

$$\mathbb{K}G = A_1 \times \cdots \times A_r.$$

Each A_i is a simple ideal, and the set of the A_i 's is uniquely determined as the set of simple ideals of $\mathbb{K}G$. The A_i 's are called the **block ideals of $\mathbb{K}G$** . Each A_i

is generated by a central primitive idempotent $e \in \mathbf{Z}(\mathbb{K}G)$. By Wedderburn-Artin theory, each A_i is isomorphic to a matrix ring over a division ring.

We now relate the above decomposition to the complex irreducible characters of G . Recall that the **Schur index** of $\chi \in \text{Irr } G$ over \mathbb{K} is the smallest positive integer $m = m_{\mathbb{K}}(\chi)$ such that $m\chi$ is afforded by a representation with entries in $\mathbb{K}(\chi)$, the field generated by \mathbb{K} and the values of χ .

2.1. Lemma. *Let $\chi \in \text{Irr } G$.*

- (i) *There is a unique block ideal A of $\mathbb{K}G$ such that $\chi(A) \neq 0$.*
- (ii) *Let $\psi \in \text{Irr } G$. Then $\psi(A) \neq 0$ if and only if ψ and χ are Galois conjugate over \mathbb{K} , that is, $\psi = \chi^\alpha$ for some $\alpha \in \text{Gal}(\mathbb{K}(\chi)/\mathbb{K})$.*
- (iii) *Write $A \cong \mathbf{M}_n(D)$ for some division ring D . Then $\mathbf{Z}(A) \cong \mathbf{Z}(D) \cong \mathbb{K}(\chi)$.*
- (iv) *$|D : \mathbf{Z}(D)| = m_{\mathbb{K}}(\chi)^2$ and $\chi(1) = nm_{\mathbb{K}}(\chi)$.*

Proof. This is standard [8, Theorems 38.1 and 38.15]. \square

It follows from Lemma 2.1 that A is itself a division ring if and only if $\chi(1) = m_{\mathbb{K}}(\chi)$. In this case, the projection $\mathbb{K}G \rightarrow A$ defines a homomorphism φ from G into the multiplicative group of D . Notice also that $\text{Ker}(\varphi) = \text{Ker}(\chi)$. For this reason, we call a character χ **skew-linear** (over \mathbb{K}), if $\chi(1) = m_{\mathbb{K}}(\chi)$. Thus skew-linear characters generalize linear characters. Since $m_{\mathbb{C}}(\chi) = 1$ for all χ , skew-linear over \mathbb{C} is the same as linear.

If $\chi \in \text{Irr}(G)$ is linear, then (trivially) the reduction to any subgroup is irreducible and linear. This fact generalizes to skew-linear characters as follows:

2.2. Lemma. *Let $\chi \in \text{Irr}(G)$ be skew-linear over the field \mathbb{K} , and $H \leq G$. Then the irreducible constituents of χ_H are skew-linear over \mathbb{K} , and are Galois conjugate over the field $\mathbb{K}(\chi)$.*

Proof. Let $\vartheta \in \text{Irr}(H)$ be a constituent of χ_H . Then [9, Lemma 10.4]

$$m_{\mathbb{K}}(\chi) \text{ divides } [\chi_H, \vartheta] |\mathbb{K}(\chi, \vartheta) : \mathbb{K}(\chi)| m_{\mathbb{K}}(\vartheta).$$

Let $\sigma \in \text{Gal}(\mathbb{K}(\chi, \vartheta)/\mathbb{K}(\chi))$. Then $[\chi_H, \vartheta^\sigma] = [\chi_H, \vartheta]$. Thus each of the $|\mathbb{K}(\chi, \vartheta) : \mathbb{K}(\chi)|$ characters ϑ^σ occurs in χ_H with multiplicity $[\chi_H, \vartheta]$. It follows that

$$\begin{aligned} [\chi_H, \vartheta] |\mathbb{K}(\chi, \vartheta) : \mathbb{K}(\chi)| \vartheta(1) &\leq \chi(1) = m_{\mathbb{K}}(\chi) \\ &\leq [\chi_H, \vartheta] |\mathbb{K}(\chi, \vartheta) : \mathbb{K}(\chi)| m_{\mathbb{K}}(\vartheta). \end{aligned}$$

This implies that equality holds throughout, in particular, $\vartheta(1) = m_{\mathbb{K}}(\vartheta)$ and $\chi_H = [\chi_H, \vartheta] \sum \vartheta^\sigma$, the sum running over $\sigma \in \text{Gal}(\mathbb{K}(\chi, \vartheta)/\mathbb{K}(\chi))$. \square

In the rest of this section, we record some (mostly well known) facts about Schur indices and blocks of group algebras for later reference.

Recall that

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$$

is the central primitive idempotent in $\mathbb{C}G$ corresponding to $\chi \in \text{Irr } G$. The following simple observation will sometimes be useful. Notice that it provides an alternative proof of $\mathbf{Z}(A) \cong \mathbb{K}(\chi)$.

2.3. Lemma. *Let $\chi \in \text{Irr } G$ and let A be the block ideal of $\mathbb{K}G$ such that $\chi(A) \neq 0$. Then*

$$A \cong \mathbb{K}(\chi)Ge_\chi \quad \text{by} \quad A \ni a \mapsto ae_\chi.$$

Proof. Set

$$e := \sum_{\alpha \in \text{Gal}(\mathbb{K}(\chi)/\mathbb{K})} e_{\chi^\alpha}.$$

We claim that $A = \mathbb{K}Ge$. We can decompose 1 into a sum of primitive idempotents in $\mathbf{Z}(\mathbb{K}G)$, and then decompose further in $\mathbf{Z}(\mathbb{C}G)$. Thus there is a unique primitive idempotent f in $\mathbf{Z}(\mathbb{K}G)$ such that $fe_\chi = e_\chi$. But then also $fe_{\chi^\alpha} = e_{\chi^\alpha}$ for all $\alpha \in \text{Gal}(\mathbb{K}(\chi)/\mathbb{K})$ and thus $fe = e$. On the other hand, $e_\chi \in \mathbb{K}(\chi)$ and $e \in \mathbb{K}G$, and thus $f = e$. This shows that $A = \mathbb{K}Ge$ as claimed.

For $\alpha \in \text{Gal}(\mathbb{K}(\chi)/\mathbb{K})$,

$$b = \sum_g b_g g \in \mathbb{K}(\chi)Ge_\chi \quad \text{implies} \quad b^\alpha := \sum_g b_g^\alpha g \in \mathbb{K}(\chi)Ge_{\chi^\alpha}.$$

Using this, it is straightforward to check that

$$\mathbb{K}(\chi)Ge_\chi \ni b \mapsto \sum_{\alpha \in \text{Gal}(\mathbb{K}(\chi)/\mathbb{K})} b^\alpha \in \mathbb{K}Ge$$

yields the inverse of the map $a \mapsto ae_\chi$. □

Since we will often have to consider characters of direct products of groups, and the corresponding blocks of the group algebra, we record the following for later reference.

2.4. Lemma. *Let $G = U \times V$ be a direct product of groups, $\sigma \in \text{Irr } U$ and $\tau \in \text{Irr } V$. Then $\chi = \sigma \times \tau \in \text{Irr } G$ and $\mathbb{K}(\chi) = \mathbb{K}(\sigma, \tau)$. Let $A_{\mathbb{K}}(\chi)$ be the block ideal of $\mathbb{K}G$ corresponding to χ , and $A_{\mathbb{K}}(\sigma)$ and $A_{\mathbb{K}}(\tau)$ the block ideals of $\mathbb{K}U$ and $\mathbb{K}V$ corresponding to σ and τ . Then*

$$A_{\mathbb{K}}(\chi) \cong \left(A_{\mathbb{K}}(\sigma) \otimes_{\mathbb{K}(\sigma)} \mathbb{K}(\chi) \right) \otimes_{\mathbb{K}(\chi)} \left(A_{\mathbb{K}}(\tau) \otimes_{\mathbb{K}(\tau)} \mathbb{K}(\chi) \right).$$

Proof. The irreducible characters of $U \times V$ are exactly the characters of the form $\chi = \sigma \times \tau$, with $\sigma \in \text{Irr } U$ and $\tau \in \text{Irr } V$ [9, Theorem 4.21]. Since $\chi((u, 1)) = \sigma(u)\tau(1)$ for $u \in U$ and similarly $\chi((1, v)) = \sigma(1)\tau(v)$ for $v \in V$, we see that $\mathbb{K}(\chi) = \mathbb{K}(\sigma, \tau)$.

Set $\mathbb{L} = \mathbb{K}(\chi)$. The natural isomorphism

$$\mathbb{L}U \otimes_{\mathbb{L}} \mathbb{L}V \rightarrow \mathbb{L}G, \quad \sum_u a_u u \otimes \sum_v b_v v \mapsto \sum_{u,v} a_u b_v (u, v)$$

sends $e_\sigma \otimes e_\tau$ to e_χ and thus induces an isomorphism

$$\mathbb{L}Ue_\sigma \otimes_{\mathbb{L}} \mathbb{L}Ve_\tau \cong \mathbb{L}Ge_\chi$$

(by comparing dimensions). By Lemma 2.3, the right hand side is isomorphic to $A_{\mathbb{K}}(\chi)$, and on the left hand side we have

$$\mathbb{L}Ue_\sigma \cong \mathbb{K}(\sigma)Ue_\sigma \otimes_{\mathbb{K}(\sigma)} \mathbb{L} \cong A_{\mathbb{K}}(\sigma) \otimes_{\mathbb{K}(\sigma)} \mathbb{L},$$

and similarly for the other factor. The result follows. \square

In Section 6, we need several deep facts about Schur indices, which we collect now. For a prime q , we write $m_q(\chi) := m_{\mathbb{Q}_q}(\chi)$, where \mathbb{Q}_q denotes the field of q -adic numbers. Sometimes, it will be convenient to use this notation also for the “infinite prime”, that is, $m_\infty(\chi) := m_{\mathbb{R}}(\chi)$.

2.5. Lemma. *Let $\chi \in \text{Irr}(G)$.*

- (i) $m_{\mathbb{Q}}(\chi)$ is the least common multiple of the local indices $m_q(\chi)$, where q runs through all primes, including the infinite one. [12, (32.19)]
- (ii) $m_{\mathbb{R}}(\chi)$ and $m_2(\chi)$ divide 2, and $m_q(\chi)$ divides $q - 1$ for odd q . [15, Theorem 4.3, Corollary 5.5]
- (iii) Let φ be an irreducible Brauer character for the prime q , and $d_{\chi\varphi}$ the decomposition number. Then $m_q(\chi)$ divides $d_{\chi\varphi}|\mathbb{Q}_q(\chi, \varphi) : \mathbb{Q}_q(\chi)|$. [4, Theorem IV.9.3]
- (iv) If the finite prime q does not divide $|G|$, then $m_q(\chi) = 1$. [4, Corollary IV.9.5]

2.6. Corollary. *Let $\chi \in \text{Irr}(G)$ with $\chi(1) = m_q(\chi)$, where q is a prime number. If $H \leq G$ is not divisible by q , then any constituent of χ_H is linear.*

Proof. This is immediate from Lemma 2.2 and Lemma 2.5 (iv). \square

3. THE NONIDEAL KERNEL

For every field \mathbb{K} and any finite group G , we define

$$\text{NKer}_{\mathbb{K}}(G) := \bigcap \{ \text{Ker}(\chi) \mid \chi(1) > m_{\mathbb{K}}(\chi) \}.$$

If $m_{\mathbb{K}}(\chi) = \chi(1)$ for every $\chi \in \text{Irr}(G)$, we set $\text{NKer}_{\mathbb{K}}(G) := G$. We call $\text{NKer}_{\mathbb{K}}(G)$ the **nonideal kernel** of G over \mathbb{K} . Notice that $\text{NKer}_{\mathbb{K}}(G)$, for any field \mathbb{K} , is characteristic in G .

3.1. Lemma. *Let $\mathbb{K} \subseteq \mathbb{L}$ be fields. Then $\text{NKer}_{\mathbb{L}}(G) \subseteq \text{NKer}_{\mathbb{K}}(G)$.*

Proof. Since $m_{\mathbb{L}}(\chi)$ divides $m_{\mathbb{K}}(\chi)$ for any $\chi \in \text{Irr } G$, any character which is skew-linear over \mathbb{L} , is also skew-linear over \mathbb{K} . The result follows. \square

3.2. Lemma. *Let G be a nonabelian group. Then*

$$\bigcap_{\substack{\chi \in \text{Irr } G \\ \chi(1) > 1}} \text{Ker } \chi = \{1\}.$$

Proof. Suppose that $g \neq 1$ is contained in the kernel of all nonlinear characters. Then, by the second orthogonality relation [9, (2.18)],

$$0 = \sum_{\chi \in \text{Irr } G} \chi(1)\chi(g) = \sum_{\substack{\chi \in \text{Irr } G \\ \chi(1) > 1}} \chi(1)^2 + \sum_{\chi \in \text{Lin } G} \chi(1)\chi(g).$$

The second sum runs over the irreducible characters of G/G' and has value $|G : G'|$ or 0, according to whether $g \in G'$ or not. It follows that the first sum must be empty. Thus G has no nonlinear characters, which means that G is abelian, as claimed. \square

3.3. Corollary. *Let G be a nonabelian group. Then $\text{NKer}_{\mathbb{C}}(G) = 1$.*

Let us say that a character α (not necessarily irreducible) is **strictly nonideal**, if $[\alpha, \chi] < \chi(1)$ for all $\chi \in \text{Irr } G$. (Such a character is afforded by a left ideal of the group algebra, which does not contain any nonzero two-sided ideal.) If at the same time, α is the character of a representation with entries in \mathbb{K} , then $m_{\mathbb{K}}(\chi)$ divides $[\alpha, \chi]$ for all $\chi \in \text{Irr } G$ [9, Corollary 10.2(c)]. Thus no constituent of α can be skew-linear over \mathbb{K} . Conversely, if S is a set of non-skew-linear characters over \mathbb{K} , then we may add the characters of the corresponding irreducible representations over \mathbb{K} and get a strictly nonideal character α which is afforded by a \mathbb{K} -representation. Since $\text{Ker } \alpha = \bigcap \text{Ker } \chi$, where χ runs through the constituents of α , it follows that every group G has a strictly nonideal character α with $\text{Ker } \alpha = \text{NKer}_{\mathbb{K}}(G)$, and such that α is afforded by a representation over \mathbb{K} . (In the case where $G = \text{NKer}_{\mathbb{K}}(G)$, the only such character is $\alpha = 0$, however.)

3.4. Lemma. *Let $H \leq G$ with $N := \text{NKer}_{\mathbb{K}}(H) < H$. Then*

$$\text{NKer}_{\mathbb{K}}(G) \leq \bigcap_{g \in G} N^g \leq \text{NKer}_{\mathbb{K}}(H).$$

Proof. Let α be a strictly nonideal character of H with $N = \text{Ker } \alpha$ and which is afforded by a representation over \mathbb{K} . Then $0 \neq \alpha^G$ is afforded by a representation over \mathbb{K} and has kernel $\bigcap_{g \in G} N^g$ [9, Lemma 5.11].

Let ρ_G be the regular character of G . Notice that a character β is strictly nonideal if and only if $\rho_G - \beta$ is a character and $[\rho_G - \beta, \chi] > 0$ for all $\chi \in \text{Irr } G$. Since $\rho_G = (\rho_H)^G$, we have that $\rho_G - \alpha^G = (\rho_H - \alpha)^G$ is a character, and

$$[\rho_G - \alpha^G, \chi] = [(\rho_H - \alpha)^G, \chi] = [\rho_H - \alpha, \chi_H]_H > 0$$

for all $\chi \in \text{Irr } G$. Thus α^G is strictly nonideal. \square

3.5. Lemma. *Let N be a normal subgroup of G , and set*

$$e_N = \frac{1}{|N|} \sum_{n \in N} n.$$

Then $\mathbb{K}Ge_N \cong \mathbb{K}[G/N]$. If $\chi \in \text{Irr } G$, then $\chi(e_N) \neq 0$ if and only if $N \leq \text{Ker}(\chi)$.

Proof. This is well known: The canonical epimorphism $\mathbb{K}G \rightarrow \mathbb{K}[G/N]$ is split by the map sending a coset Ng to $(1/|N|) \sum Ng = e_N g$. This proves the first statement.

If $N \leq \text{Ker}(\chi)$, then any representation affording χ sends e_N to the identity map. If $N \not\leq \text{Ker}(\chi)$, then any representation affording χ must send e_N to 0. \square

3.6. Lemma. *Let $N := \text{NKer}_{\mathbb{K}}(G)$, and e_N as in Lemma 3.5. Then $\mathbb{K}G(1 - e_N)$ is a direct product of division rings. In particular, every idempotent $f \in \mathbb{K}G$ with $f e_N = 0$ is central.*

Proof. By Lemma 3.5, it follows that $\mathbb{K}G(1 - e_N)$ is the direct product of the block ideals which correspond to $\chi \in \text{Irr } G$ with $N \not\leq \text{Ker}(\chi)$. By definition of N , any such χ is skew-linear over \mathbb{K} , and thus the corresponding block ideal is a division ring.

In a direct product of division rings, every idempotent is central. \square

Proof of Lemma A. The first part of Lemma A is contained in Lemma 3.6. Conversely, if $\mathbb{K}G(1 - e_N)$ is a direct product of division rings, then the above considerations yield that when $m_{\mathbb{K}}(\chi) < \chi(1)$, we must have $N \subseteq \text{Ker}(\chi)$, and thus $N \leq \text{NKer}_{\mathbb{K}}(G)$. \square

Following Blackburn [3], for any group G , we set

$$\mathbf{R}(G) := \bigcap \{U \leq G \mid U \text{ not normal in } G\}.$$

If every subgroup of G is normal, then we set $\mathbf{R}(G) = G$. Blackburn [3] classified finite groups in which $\mathbf{R}(G) \neq 1$. Therefore, a group G with $\mathbf{R}(G) \neq 1$ is called a **Blackburn group**. The following result shows why this is relevant for us:

3.7. Lemma. *For any finite group G and field \mathbb{K} of characteristic zero, we have $\text{NKer}_{\mathbb{K}}(G) \leq \mathbf{R}(G)$.*

Proof. Suppose $U \leq G$ is such that $N := \text{NKer}_{\mathbb{K}}(G) \not\leq U \leq G$. We need to show that $U \not\trianglelefteq G$.

Set $f = (1 - e_N)e_U$, with e_N as before, and analogously $e_U := (1/|U|) \sum_{u \in U} u$. Then $f^2 = f \in \mathbb{K}G(1 - e_N)$, since e_N is central in $\mathbb{K}G$. Thus f is central in $\mathbb{K}G$ by Lemma 3.6. We compute

$$f = \frac{1}{|U|} \sum_{u \in U} u - \frac{1}{|NU|} \sum_{x \in NU} x.$$

As N is not contained in U , we have $U < NU$. As $g^{-1}fg = f$ for all $g \in G$, it follows that $U \trianglelefteq G$. \square

4. DEDEKIND GROUPS

In this section, we compute $\text{NKer}_{\mathbb{K}}(G)$ for Dedekind groups, and determine when $\mathbb{K}G$ is a direct product of division rings. These results are mostly known.

Recall that a **Dedekind group** is a finite groups in which all subgroups are normal. First, we recall Dedekind's classification of these groups [7, Satz III.7.12 on p. 308].

4.1. Theorem (Dedekind 1897). *Let G be a finite group, such that every subgroup of G is normal. Then either G is abelian, or*

$$G \cong Q_8 \times (C_2)^r \times A \quad (r \geq 0),$$

where A is abelian of odd order.

Let $\tau \in \text{Irr}(Q_8)$ be the irreducible, faithful character of degree 2. Then $\mathbb{H} := \mathbb{Q}Q_8e_\tau$ is a division ring, the rational quaternions. \mathbb{H} can also be described as the \mathbb{Q} -vector space with basis $\{1, i, j, k\}$ and multiplication defined by $i^2 = j^2 = -1$, $k = ij = -ji$.

4.2. Theorem. *Let \mathbb{K} be a field and G be a group. Then $\mathbb{K}G$ is a direct product of division rings if and only if either G is abelian, or $G \cong Q_8 \times (C_2)^r \times A$, where A is abelian of odd order, and $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{K}(\lambda)$ is a division ring for all $\lambda \in \text{Lin}(A)$.*

Proof. Suppose $\mathbb{K}G$ is a direct product of division rings. Then all subgroups of G are normal in G by Lemma 3.7 (as $\text{NKer}_{\mathbb{K}}(G) = G$, or directly from the argument in the proof of Lemma 3.7). It follows that either G is abelian, or $G \cong Q_8 \times (C_2)^r \times A$ with A abelian of odd order.

In the second case, let $\tau \in \text{Irr}(Q_8)$ be the irreducible, faithful character of degree 2. Then

$$\mathbb{K}Q_8e_\tau \cong \mathbb{H} \otimes_{\mathbb{Q}} \mathbb{K},$$

the quaternions over \mathbb{K} . Any nonlinear, irreducible character of $G = Q_8 \times (C_2)^r \times A$ has the form $\chi = \tau \times \sigma \times \lambda$, where $\sigma \in \text{Lin}(C_2)^r$ and $\lambda \in \text{Lin } A$. The corresponding block ideal of the rational group algebra is, by Lemma 2.4, isomorphic to

$$\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{K}(\lambda).$$

The result follows. □

4.3. Remark. If $G \cong Q_8 \times (C_2)^r \times A$ with A abelian of odd order, then either $\mathbb{K}G$ is a direct product of division rings, or $\text{NKer}_{\mathbb{K}}(G) = 1$.

Proof. Suppose that $\mathbb{K}G$ is not a direct product of division rings. Then there is some $\lambda \in \text{Lin}(A)$ such that $\mathbb{K}(\lambda)$ is a splitting field for \mathbb{H} . As before, let $\tau \in \text{Irr}(Q_8)$ be the faithful irreducible character of Q_8 . Then $\text{Ker}(\tau \times 1 \times \lambda) = 1 \times (C_2)^r \times \text{Ker}(\lambda)$.

It follows that $\text{NKer}_{\mathbb{K}}(G) \subseteq 1 \times \text{Ker}(\mu)$ for every $\mu \in \text{Lin}((C_2)^r \times A)$ such that $\text{ord}(\lambda)$ divides the order of μ . Since A contains elements of order $\text{ord}(\lambda)$, we see that $\text{NKer}_{\mathbb{K}}(G) = 1$. □

Notice that for a linear character λ , we have $\mathbb{K}(\lambda) = \mathbb{K}(\varepsilon_n)$, where ε_n is a primitive n -th root of unity and $n = \text{ord}(\lambda)$. The following lemma collects some results. These will be needed also in the proof of Theorem D.

4.4. Lemma.

- (i) $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{K}$ is a division ring if and only if -1 is not a sum of two squares in \mathbb{K} .
- (ii) $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_2$ and $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{R}$ are division rings, and $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for p odd is not a division ring. (Here \mathbb{Q}_p is the field of p -adic numbers.)

Let ε_n be a primitive n -th root of unity, where n is odd. Then

- (iii) $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}(\varepsilon_n)$ is a division ring if and only if the multiplicative order of 2 in $(\mathbb{Z}/n\mathbb{Z})^*$ is odd, if and only if $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_2(\varepsilon_n)$ is a division ring.
- (iv) $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}, \varepsilon_n)$ is a division ring only when $n = 1$.

Proof. (i) and (ii) are well known [8, Example 38.13(a)] [14, Ch. III, Théorème 1]. Assertion (iii) is a result of Moser [11]. (This can be shown without using the Hasse-Minkowski principle: If the residue class of 2 in $(\mathbb{Z}/p)^*$ has even multiplicative order $2r$, then $2^r \equiv -1 \pmod{p}$, and thus p divides $2^r + 1$. Then an elementary argument shows that -1 is a sum of two squares in $\mathbb{Q}(\varepsilon_p)$ [8, Example 38.13(d)]. If 2 has odd order in $(\mathbb{Z}/n\mathbb{Z})^*$, then $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_2(\varepsilon_n)$ is also a division ring.)

To see (iv), assume that $n > 1$. We have to show that -1 is a sum of two squares in $\mathbb{K} := \mathbb{Q}(\sqrt{2}, \varepsilon_n)$. By the Hasse-Minkowski principle, it suffices to show that -1 is a square in each possible completion of \mathbb{K} . Since $n > 1$, \mathbb{K} can not be embedded into \mathbb{R} . If p is odd, then -1 is a sum of two squares in \mathbb{Q}_p already. Finally, $\mathbb{Q}_2(\sqrt{2})$ is a quadratic extension of \mathbb{Q}_2 and thus a splitting field of \mathbb{H} [10, Lemma VI.2.14]. (We notice that in (iv), we can replace $\mathbb{Q}(\sqrt{2})$ by any field such that the completions at all prime ideals over 2 yield extensions of *even* degree over \mathbb{Q}_2 .) \square

As a consequence, we get the following results.

4.5. Theorem (Sehgal 1975 [13]). *The group algebra $\mathbb{Q}G$ is a direct product of division rings if and only if one of the following holds:*

- (i) G is abelian.
- (ii) $G \cong Q_8 \times (C_2)^r \times A$, where $r \geq 0$, and A is abelian of odd order, and the multiplicative order of 2 in $(\mathbb{Z}/|A|)^*$ is odd.

4.6. Theorem.

- (i) $\mathbb{Q}_2 G$ is a direct product of division rings if and only if $\mathbb{Q}G$ is a direct product of division rings.
- (ii) Let p be an odd prime. Then $\mathbb{Q}_p G$ is a direct product of division rings if and only if G is abelian.

- (iii) $\mathbb{R}G$ is a direct product of division rings if and only if either G is abelian, or $G \cong Q_8 \times (C_2)^r$ for some $r \geq 0$.

5. CLASSIFICATION OVER THE REALS

In this section, we prove Theorem B. We begin with the (maybe more interesting) “only if” part.

5.1. Lemma. *Suppose that $\text{NKer}_{\mathbb{R}}(G) \neq 1$, and $\langle g \rangle \not\trianglelefteq G$. Then g has order 4, and $\text{NKer}_{\mathbb{R}}(G) = \langle g^2 \rangle$ has order 2.*

Proof. By Lemma 3.7 and the definition of $\mathbf{R}(G)$, we have

$$1 \neq N := \text{NKer}_{\mathbb{R}}(G) \leq \mathbf{R}(G) < \langle g \rangle.$$

The last inequality is strict since $\mathbf{R}(G)$ is normal in G , but $\langle g \rangle$ is not. In particular, the first claim of the lemma implies the second one.

Let $\lambda \in \text{Lin}\langle g \rangle$ be faithful. By Lemma 3.5 applied to $N \trianglelefteq \langle g \rangle$ and since $N \neq 1$, it follows $\lambda(e_N) = 0$. Thus

$$f = e_\lambda + e_{\bar{\lambda}} = \frac{1}{|\langle g \rangle|} \sum_{h \in \langle g \rangle} (\bar{\lambda}(h) + \lambda(h))h \in \mathbb{R}\langle g \rangle$$

is an idempotent with $fe_N = 0$. It follows from Lemma 3.6 that f is a central idempotent in $\mathbb{R}G$, and so $f^x = f$ for all $x \in G$. But by assumption, there is some $x \in G$ such that $g^x \notin \langle g \rangle$. It follows that $\bar{\lambda}(g) + \lambda(g) = 0$. As $\lambda(g)$ is an n -th root of unity, where $n = \text{ord}(g)$, this is only possible when $\text{ord}(g) = 4$. \square

5.2. Lemma. *Suppose that $1 < \text{NKer}_{\mathbb{R}}(G) < G$ and that G is not a 2-group. Then G is generalized dicyclic.*

Proof. By Lemma 5.1, we have that $\text{NKer}_{\mathbb{R}}(G) = \mathbf{R}(G) = \langle z \rangle$, where z has order 2. Every odd-order subgroup of G is normal in G , and in particular the Sylow p -subgroups, for p odd, generate a normal 2-complement, U , of G . As U is Dedekind, it follows that U is abelian.

Now set $A = \mathbf{C}_G(U)$, which contains U . There is $g \in G$ such that $\langle g \rangle \not\trianglelefteq G$. By Lemma 5.1, we have $g^4 = 1$. If $gu = ug$ for some $u \in U$, then $\langle g \rangle$ is characteristic in $\langle gu \rangle = \langle g \rangle \times \langle u \rangle$, and thus $\langle gu \rangle \not\trianglelefteq G$. Again by Lemma 5.1, it follows that $(gu)^4 = 1$ and thus $u = 1$. Thus $\mathbf{C}_U(g) = 1$ and $g \notin A$. In particular, $A < G$.

Conversely, let $g \notin A$, and let $s = g_2$ be the 2-part of g . Then $gA = sA$ and thus $s \notin A$. Thus $u^s \neq u$ for some $u \in U$, and thus $s^u = s[s, u] \notin \langle s \rangle$. It follows that $\langle s \rangle$ is not normal in G , and thus $\langle g \rangle$ is not normal in G . By Lemma 5.1, it follows that $g^2 = z$ (and $s = g$).

In particular, for $g \in G \setminus A$ and $a \in A$, we have $g^2 = z = (ga)^2 = g^2 a^g a$, and thus $a^g = a^{-1}$. For $u \in U$ and $g, h \in G \setminus A$ we have $u^g = u^{-1} = u^h$ and thus $gh^{-1} \in \mathbf{C}_G(U) = A$, so $|G : A| = 2$. Thus G is generalized dicyclic. \square

To finish the proof of the “only if” part of Theorem B, we use a part of Blackburn’s classification [3, Theorem 1]:

5.3. Theorem (Blackburn 1966). *Let G be a p -group with $\mathbf{R}(G) \neq 1$. Then one of the following holds:*

- (i) G is abelian.
- (ii) $p = 2$ and G is generalized dicyclic.
- (iii) $p = 2$ and $G \cong C_4 \times Q_8 \times (C_2)^r$, $r \in \mathbb{N}$.
- (iv) $p = 2$ and $G \cong Q_8 \times Q_8 \times (C_2)^r$, $r \in \mathbb{N}$.

Using Theorem 5.3, it is rather straightforward to determine all finite groups G with $\mathbf{R}(G) \neq 1$, but a rather long list emerges [3, Theorem 2]. However, due to Lemma 5.2, we do not need to go through the longer list of finite groups with $\mathbf{R}(G) \neq 1$.

Proof of Theorem B, “only if”. Suppose that $\mathrm{NKer}_{\mathbb{R}}(G) \neq 1$. If $\mathrm{NKer}_{\mathbb{R}}(G) = G$, then G is abelian or $G \cong Q_8 \times (C_2)^r$, by Theorem 4.6.

If $1 < \mathrm{NKer}_{\mathbb{R}}(G) < G$ and G is not a 2-group, then G is generalized dicyclic, by Lemma 5.2. If G is a 2-group, then it follows from Blackburn’s classification of 2-groups with $\mathbf{R}(G) \neq 1$ (Theorem 5.3) that G appears on the list in Theorem B. \square

We now show that conversely, the groups appearing in Theorem B all have $\mathrm{NKer}_{\mathbb{R}}(G) \neq 1$. To show that certain characters are skew-linear, we use the Frobenius-Schur indicator. Recall that for $\chi \in \mathrm{Irr} G$, its **Frobenius-Schur indicator** is defined by

$$\nu_2(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

When $\nu_2(\chi) = 1$, then $\chi = \bar{\chi}$ and χ is afforded by a representation with entries in \mathbb{R} , so $m_{\mathbb{R}}(\chi) = 1$. When $\nu_2(\chi) = 0$, then $\chi \neq \bar{\chi}$, and again $m_{\mathbb{R}}(\chi) = 1$. Finally, when $\nu_2(\chi) = -1$, then $\chi = \bar{\chi}$, but $m_{\mathbb{R}}(\chi) = 2$. In the last case, there is a simple $\mathbb{R}G$ -module affording 2χ , and $\mathrm{End}_{\mathbb{R}G}(S) \cong \mathbb{H}$, the division ring of Hamilton’s quaternions [8, Theorem 13.12].

In particular, $\chi \in \mathrm{Irr} G$ is skew-linear over \mathbb{R} , if and only if either $\chi(1) = 1$ (χ is linear), or $\chi(1) = 2$ and $\nu_2(\chi) = -1$.

We begin by considering generalized dicyclic groups.

5.4. Lemma. *Let G be generalized dicyclic, and let $g \in G$ and $A \trianglelefteq G$ be as in the definition. Then $\mathbf{R}(G) = \mathrm{NKer}_{\mathbb{R}}(G) = G$ if $G/\langle g^2 \rangle$ is abelian, and $\mathbf{R}(G) = \mathrm{NKer}_{\mathbb{R}}(G) = \langle g^2 \rangle$ else.*

Proof. First, observe that $g^2 = (g^2)^g = g^{-2}$ and thus $g^4 = 1$. Moreover, for any $a \in A$, we have $(ga)^2 = g^2 a^g a = g^2$. By assumption, $g^2 \neq 1$.

In view of Lemma 5.1, it suffices to show that $\langle g^2 \rangle \subseteq \mathrm{NKer}_{\mathbb{R}}(G)$, that is, all characters $\chi \in \mathrm{Irr} G$ with $g^2 \notin \mathrm{Ker} \chi$ are skew-linear. (In the case when $G/\langle g^2 \rangle$ is

abelian, all characters of $G/\langle g^2 \rangle$ are linear and thus it will follow that all characters of G are skew-linear and G is Dedekind. Conversely, if $\text{NKer}_{\mathbb{R}}(G) > \langle g^2 \rangle$, then $\text{NKer}_{\mathbb{R}}(G) = G$ by Lemma 5.1, and then $G \cong Q_8 \times (C_2)^r$ by Theorem 4.6 and $G/\langle g^2 \rangle$ is abelian.)

So suppose that $\chi \in \text{Irr } G$ is not linear, and $g^2 \notin \text{Ker } \chi$. Let $\lambda \in \text{Lin } A$ be a constituent of the restriction χ_A . Then $\chi = \lambda^G$ by Clifford theory [9, Corollary 6.19]. As $a^g = a^{-1}$ for all $a \in A$, we have $\lambda^g = \bar{\lambda}$. Also, $\lambda(g^2) \neq 1$, and thus $\lambda(g^2) = -1$ and $\chi(g^2) = -2$. It follows that

$$\begin{aligned} \nu_2(\chi) &= \frac{1}{|G|} \sum_{x \in G} \chi(x^2) = \frac{1}{|G|} \sum_{a \in A} (\chi((ga)^2) + \chi(a^2)) \\ &= \frac{1}{|G|} \left(-2|A| + \sum_{a \in A} (\lambda(a^2) + \bar{\lambda}(a^2)) \right) \\ &= \frac{-2|A|}{|G|} = -1. \end{aligned}$$

Here we have used that $(ga)^2 = g^2$ for all $a \in A$, and that $\sum_{a \in A} \lambda(a^2) = \sum_{a \in A} \lambda^2(a) = 0$ since $\bar{\lambda} \neq \lambda$ and thus $\lambda^2 \neq 1$. Since $\nu_2(\chi) = -1$ and $\chi(1) = 2$, it follows that χ is indeed skew-linear, as claimed. \square

5.5. Lemma. *When $G = \langle u \rangle \times \langle x, y \rangle \times E$ with $\langle u \rangle \cong C_4$, $\langle x, y \rangle \cong Q_8$ and $E \cong (C_2)^r$, then $\text{NKer}_{\mathbb{R}}(G) = \mathbf{R}(G) = \langle u^2 x^2 \rangle \neq 1$.*

Proof. As $\langle ux \rangle \not\trianglelefteq G$, we have $\mathbf{R}(G) \leq \langle u^2 x^2 \rangle$. Let τ be the nonlinear irreducible character of $\langle x, y \rangle$ and λ a character of $\langle u \rangle$ with $\lambda \neq \bar{\lambda}$. If χ is a character with $\chi(u^2 x^2) \neq \chi(1)$, then either χ is linear, or $\chi = \lambda^2 \times \tau \times \sigma$, $\sigma \in \text{Lin } E$. The latter characters all have $\nu_2(\chi) = -1$. Thus $\langle u^2 x^2 \rangle \subseteq \text{NKer}_{\mathbb{R}}(G)$. \square

5.6. Lemma. *When $G = \langle u, v \rangle \times \langle x, y \rangle \times E$ with $\langle u, v \rangle \cong \langle x, y \rangle \cong Q_8$ and $E \cong (C_2)^r$, then $\text{NKer}_{\mathbb{R}}(G) = \mathbf{R}(G) = \langle u^2 x^2 \rangle \neq 1$.*

Proof. As $\langle ux \rangle \not\trianglelefteq G$, we have $\mathbf{R}(G) \leq \langle u^2 x^2 \rangle$. Let τ_1 and τ_2 be the nonlinear characters of $\langle u, v \rangle$ and $\langle x, y \rangle$, respectively. If $\chi(u^2 x^2) \neq \chi(1)$, then either $\chi = \tau_1 \times \lambda \times \sigma$ with $\lambda \in \text{Lin} \langle x, y \rangle$ and $\sigma \in \text{Lin}(E)$, or $\chi = \lambda \times \tau_2 \times \sigma$ with $\lambda \in \text{Lin} \langle u, v \rangle$ and $\sigma \in \text{Lin}(E)$. In both cases, $\nu_2(\chi) = -1$ and thus $\langle u^2 x^2 \rangle \leq \text{NKer}_{\mathbb{R}}(G)$. \square

This lemma finishes the proof of the “if” part of Theorem B.

6. CLASSIFICATION OVER THE RATIONAL NUMBERS

In this section, we prove Theorem D. Throughout, we write $\text{NKer}(G) := \text{NKer}_{\mathbb{Q}}(G) \neq 1$.

Recall that a *Blackburn group* is a finite group G such that $\mathbf{R}(G)$, the intersection of all nonnormal subgroups of G , is nontrivial. For later reference, we record the following observation (which is part of the argument used by Blackburn to classify these groups):

6.1. Lemma. *Let G be a Blackburn group and p a prime dividing $|\mathbf{R}(G)|$. Then G has a normal p -complement A such that every subgroup of A is normal in G . If A is nonabelian, then $G = Q_8 \times (C_2)^r \times H$, where H is a Blackburn group of odd order.*

Proof. By definition of $\mathbf{R}(G)$, all the Sylow q -subgroups for $q \neq p$ are normal in G , and thus generate a normal p -complement, A . By definition of $\mathbf{R}(G)$, it follows also that every subgroup of A is normal in G .

In particular, A is a Dedekind group. If A is nonabelian, then $S \in \text{Syl}_2(G)$ is isomorphic to $Q_8 \times (C_2)^r$, by Theorem 4.1. As $S \trianglelefteq G$, there is a 2-complement H . Since every subgroup of S is normal in G , it is easy to see that H centralizes S and thus $G = S \times H$ (this is also shown in [3, Proof of Theorem 2(e)]). Any nonnormal subgroup of H is nonnormal in G and thus $\mathbf{R}(G) \leq \mathbf{R}(H)$. \square

Thus A is a Dedekind group and $G = PA$ for any $P \in \text{Syl}_p(G)$. The classification of Blackburn groups can now be obtained by considering the different possibilities for P and A (using the fact that P is also a Blackburn group and Theorem 5.3 for P , and Theorem 4.1 for A). However, in our proof of Theorem D, we do not have to consider all the cases of Blackburn's classification separately. First, we reduce to the case that A is abelian.

6.2. Theorem. *Let $G = Q_8 \times (C_2)^r \times H$ with H of odd order. Then $\text{NKer}(G) \neq 1$ if and only if $\text{NKer}(H) \neq 1$ and the multiplicative order of 2 in $(\mathbb{Z}/|H|)^*$ is odd.*

Proof. In view of Theorem 4.5, we may assume that H is nonabelian. Thus $\text{NKer}(G) \leq \text{NKer}(H) \leq \mathbf{R}(H) < H$ by Lemma 3.4 and Lemma 3.7. Assume $\text{NKer}(G) \neq 1$ and let $z \in \text{NKer}(G)$ have prime order p . Let A be the abelian p -complement of H and suppose $\lambda \in \text{Lin}(\langle z, A \rangle)$ has maximal possible order. (This implies $\lambda(z) \neq 1$, in particular.) Then any $\chi \in \text{Irr}(H \mid \lambda)$ is skew-linear. For $\tau \in \text{Irr}(Q_8)$ with $\tau(1) = 2$, we must have that $\tau \times \chi$ is also skew-linear. Lemma 2.4 yields in particular, that $\mathbb{Q}(\chi)$ must not be a splitting field for \mathbb{H} (the quaternions over \mathbb{Q}).

On the other hand, we have $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\varepsilon)$, where ε is a primitive $|H|$ -th root of unity. Since every prime dividing $|H|$ also divides $\text{ord}(\lambda)$, we see that $|\mathbb{Q}(\lambda) : \mathbb{Q}(\chi_A)|$ is odd. Therefore, $|\mathbb{Q}(\varepsilon) : \mathbb{Q}(\chi)|$ is odd as well. Thus $\mathbb{Q}(\chi)$ is a splitting field for the quaternions, if and only if $\mathbb{Q}(\varepsilon)$ is a splitting field for the quaternions. Now Lemma 4.4(iii) yields that the condition on the order of 2 mod $|H|$ holds.

Conversely, assume that this condition holds, and let $\chi \in \text{Irr}(H)$ be skew-linear over \mathbb{Q} , and $\sigma \in \text{Irr}(S)$, where $S = Q_8 \times (C_2)^r$. Let D be the block ideal of $\mathbb{Q}H$ corresponding to χ . This is a division ring with center isomorphic to $\mathbb{Q}(\chi)$. If σ is linear, then the block ideal corresponding to $\sigma \times \chi$ is again isomorphic to D . If σ is nonlinear, then the block ideal corresponding to $\sigma \times \chi$ is isomorphic to

$$(\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}(\chi)) \otimes_{\mathbb{Q}(\chi)} D,$$

by Lemma 2.4. This is a division ring since both factors are division rings and one has dimension 4 over its center, and the other has odd dimension. Thus $\sigma \times \chi$ is skew-linear. This shows that $\text{NKer}(G) = \text{NKer}(H)$. The theorem follows. \square

Next, we consider nilpotent groups.

6.3. Theorem. *Let G be nilpotent. Then $\text{NKer}(G) \neq 1$ if and only if one of the following holds.*

- (i) G is abelian.
- (ii) $G = S \times A$, where $S \in \text{Syl}_2(G)$ is a nonabelian group from the list in Theorem 5.3 and has exponent 4, and A is abelian of odd order, and the multiplicative order of 2 in $(\mathbb{Z}/|A|)^*$ is odd.
- (iii) G is a generalized dicyclic 2-group.

Proof. If G is nonabelian, then the Sylow 2-subgroup S is nonabelian, and all other Sylow subgroups are abelian, by Theorem 5.3. Thus $G = S \times A$ with A abelian. If S has exponent 4, then the nonlinear, but skew-linear characters of S yield the quaternions over \mathbb{Q} as block ideal of the rational group algebra $\mathbb{Q}S$, and the result follows from Lemma 4.4(iii), and Lemma 2.4.

If S contains elements of order 8 or greater, then S is generalized dicyclic, and there is a skew-linear $\sigma \in \text{Irr } S$ such that $S/\text{Ker}(\sigma)$ is a dicyclic (=generalized quaternion) group of order at least 16. Then $\mathbb{Q}(\sigma)$ contains $\sqrt{2}$. As $S/\text{Ker}(\sigma)$ has a subgroup of order 8 isomorphic to the quaternion group, the block ideal of $\mathbb{Q}S$ corresponding to σ is isomorphic to the quaternions over a field containing $\sqrt{2}$. But since $\mathbb{H} \otimes \mathbb{Q}_2(\sqrt{2})$ splits (Lemma 4.4(iv)), the Schur index of such a character at the prime 2 is trivial. If $G = S \times A$, then any character $\sigma \times \lambda$ with $1_A \neq \lambda \in \text{Lin } A$ has trivial Schur index over the reals, and over all other primes anyway. Thus we can have $\text{NKer}(G) \neq 1$ only if $G = S$ in this case. \square

To prove Theorem D, we can now assume that the p -complement A in Lemma 6.1 is abelian, and that $G = PA$ is not nilpotent. In other words, $\mathbf{C}_P(A) < P$, where $P \in \text{Syl}_p(G)$. It is not difficult to see that P is then either abelian or generalized dicyclic: Namely, $\mathbf{R}(P) \neq 1$ and so P occurs on the list from Theorem 5.3. If $P \cong C_4 \times Q_8 \times (C_2)^r$ or $P \cong Q_8 \times Q_8 \times (C_2)^r$, however, then P is generated by elements u such that $\langle u \rangle \cap \mathbf{R}(P) = 1$ and thus P would centralize A , so this is impossible. (Alternatively, look at Blackburn's list [3, Theorem 2].)

It remains to show that in this situation, (iv) in Theorem D holds, or $p = 2$ and G is generalized dicyclic.

We begin with some elementary observations, which were also used in Blackburn's classification.

6.4. Lemma. *Let Q be a finite abelian q -group and suppose that P acts on Q by automorphisms such that every subgroup of Q is P -invariant, and $(|P|, |Q|) = 1$.*

Then $P/\mathbf{C}_P(Q)$ is cyclic of order dividing $q-1$, and $\mathbf{C}_P(x) = \mathbf{C}_P(Q) = P_\lambda$ for every $1 \neq x \in Q$ and $1_Q \neq \lambda \in \text{Lin } Q$.

Proof. Take $x \in Q$ of maximal order and $u \in P$. Since $x^u \in \langle x \rangle$ by assumption, we have $x^u = x^k$ for some $k \in \mathbb{N}$. If $y \in Q$ with $\langle x \rangle \cap \langle y \rangle = 1$, then $y^u = y^k$, since u maps $\langle y \rangle$ and $\langle xy \rangle$ to itself. It follows that $y^u = y^k$ for all $y \in Q$.

Therefore, $P/\mathbf{C}_P(Q)$ is isomorphic to a q' -subgroup of $\text{Aut}(\langle x \rangle)$, and thus is cyclic of order dividing $q-1$.

Finally, suppose $1 \neq x \in Q$ and $x^u = x$ for some $u \in P$. As we have just seen, there is $k \in \mathbb{N}$ such that $y^u = y^k$ for all $y \in Q$. It follows that $k \equiv 1 \pmod{q}$ (as $q \mid \text{ord}(x)$). Since $|P/\mathbf{C}_P(Q)|$ divides $q-1$, it follows that $k^{q-1} \equiv 1 \pmod{q^n}$, where q^n is the exponent of Q . But this yields that $k \equiv 1 \pmod{q^n}$ and thus $u \in \mathbf{C}_P(Q)$ as claimed. The proof for $\lambda \in \text{Lin } Q$ is similar, using that there is ℓ such that $\mu^u = \mu^\ell$ for all $\mu \in \text{Lin } Q$. \square

6.5. Lemma. *Let $G = PA$ be a Blackburn group with normal abelian p -complement A and $\mathbf{R}(G) \leq P \in \text{Syl}_p(G)$. Suppose that $\chi \in \text{Irr}(G)$ is skew-linear over \mathbb{Q}_q , where q is a prime dividing $|A|$. Then P centralizes every Sylow r -subgroup R of A such that $r \neq q$ and $R \not\subseteq \text{Ker}(\chi)$.*

Proof. Let $r \neq q$ and $R \in \text{Syl}_r(A)$, and assume that $R \not\subseteq \text{Ker}(\chi)$. Let $\lambda \in \text{Lin}(R)$ be a linear constituent of χ_R , so that $\lambda \neq 1$.

By Lemma 6.4, $\mathbf{C}_P(R) = \mathbf{C}_P(x)$ for any $1 \neq x \in R$, and thus also

$$\mathbf{C}_P(R) = P_\lambda := \{u \in P \mid \lambda^u = \lambda\}.$$

Consider the subgroup $H = PR$, and choose a constituent $\vartheta \in \text{Irr}(H)$ of χ_H that lies over λ . Then $\vartheta = \psi^H$ for some $\psi \in \text{Irr}(H_\lambda)$, where $H_\lambda = \mathbf{C}_P(R)R$. Thus $\vartheta(1) \geq |P : \mathbf{C}_P(R)|$. On the other hand, by Corollary 2.6 we have that $\vartheta(1) = 1$, and thus $P = \mathbf{C}_P(R)$ as claimed. \square

6.6. Lemma. *Let $G = PA$ be a group with a normal abelian p -complement A and $1 \neq \text{NKer}(G) \leq P \in \text{Syl}_p(G)$. Suppose that $|P : \mathbf{C}_P(A)| > 2$. Then P is abelian, and there is exactly one Sylow subgroup of A which is not centralized by P .*

Proof. Let $z \in \text{NKer}(G) \subseteq P$ be an element of order p . Choose $\tau \in \text{Irr}(P)$ with $z \notin \text{Ker}(\tau)$.

For $\lambda \in \text{Lin}(A)$ arbitrary, we have

$$[(\tau^G)_A, \lambda]_A = [(\tau_{P \cap A})^A, \lambda]_A = [\tau_{P \cap A}, \lambda_{P \cap A}]_{P \cap A} = \tau(1) > 0,$$

as $P \cap A = 1$. Thus for any $\lambda \in \text{Lin}(A)$, there is $\chi \in \text{Irr}(G)$ such that $[\chi, \tau^G] > 0$ and $[\chi_A, \lambda] > 0$. We apply this to a λ such that $\lambda_R \neq 1_R$ for each Sylow subgroup, R , of A . Thus there is a $\chi \in \text{Irr}(G)$ lying over τ and such that $\text{Ker}(\chi)$ contains no Sylow subgroup of A .

Notice that $\mathbf{C}_P(A) = P_\lambda$ for such a λ , by Lemma 6.4. As χ is induced from a character of G_λ , it follows that $\chi(1) \geq |G : G_\lambda| = |P : \mathbf{C}_P(A)| > 2$.

Because $z \notin \text{Ker}(\chi)$ and $z \in \text{NKer}(G)$, it follows that χ is skew-linear over \mathbb{Q} and thus $m_{\mathbb{Q}}(\chi) = \chi(1)$. By Ito's theorem [9, Theorem 6.15], $\chi(1)$ divides $|G : A| = |P|$ and thus is a power of p . It follows from Lemma 2.5(i) that there is a prime q (possibly infinite) such that $m_q(\chi) = m_{\mathbb{Q}}(\chi) = \chi(1)$.

Since $\chi(1) \geq |P : \mathbf{C}_P(A)| > 2$, it follows from Lemma 2.5(ii) that the prime q , such that $m_q(\chi) = \chi(1)$, must be a finite, odd prime and $q \neq p$. It follows that q divides $|A|$. Now Corollary 2.6 yields that τ is linear. Since the only assumption on $\tau \in \text{Irr } P$ was that $z \notin \text{Ker}(\tau)$, Lemma 3.2 yields that P is abelian. Lemma 6.5 yields that P centralizes all Sylow subgroups of A except the Sylow q -subgroup. \square

6.7. Lemma. *Let $G = SA$ be a group, where A is a normal (abelian) 2-complement and $1 \neq \text{NKer}(G) \leq S \in \text{Syl}_2(G)$, and suppose $|S : \mathbf{C}_S(A)| = 2$. Then either G is as in Lemma 6.6 (with $p = 2$), or G is generalized dicyclic.*

(Notice that A is automatically abelian here since A is a Dedekind group of odd order.)

Proof of Lemma 6.7. First we show that $C := \mathbf{C}_S(A)$ is abelian. Let $z \in \text{NKer}(G)$ have order 2. Then $z \in \mathbf{Z}(G)$ and thus $z \in C$. If C is not abelian, there is $\tau \in \text{Irr}(C)$ with $\tau(z) \neq \tau(1) > 1$ (Lemma 3.2). Let $t \in S \setminus C$, and let $\lambda \in \text{Lin}(A)$ be such that $\lambda^t \neq \lambda$. Then $\chi := (\tau \times \lambda)^G \in \text{Irr}(G)$, and $\chi(1) \geq 2\tau(1) > 2$. By Lemma 2.5(ii), χ can not be skew-linear over \mathbb{R} or \mathbb{Q}_2 . Since χ_C has a non-linear constituent, χ can not be skew-linear over \mathbb{Q}_q for odd primes q , by Corollary 2.6. As $\chi(1) = 2^r$, it follows from Lemma 2.5(i) that $m_{\mathbb{Q}}(\chi) < \chi(1)$, and thus $z \notin \text{NKer}(G)$, contradiction. Thus C is abelian as claimed, and $G = SA$ has the abelian subgroup CA of index 2.

Fix $t \in S \setminus C$. Notice that $A = [A, t] \times C_A(t)$. Since every subgroup of A is normal in G , the factors of this decomposition have coprime orders. Also, we have $[A, t] \neq 1$ by assumption, and t inverts the elements in $[A, t]$.

Consider first the case $\mathbf{C}_A(t) \neq 1$. Pick some $\lambda \in \text{Lin}(A)$ such that $\text{Ker}(\lambda)$ contains no Sylow subgroup of A . Then $\lambda^t \notin \{\lambda, \bar{\lambda}\}$. Consider extensions μ to CA with $\mu(z) = -1$, where $z \in \text{NKer}(G)$ has order 2 as before. As $\mu^t \neq \mu$, we have $\chi = \mu^G \in \text{Irr}(G)$. Then χ remains irreducible modulo 2, and thus $m_2(\chi) = 1$, by Lemma 2.5(iii). As $\mu^t \neq \bar{\mu}$, we have also $m_{\mathbb{R}}(\chi) = 1$. But as $z \notin \text{NKer}(G)$, it follows that $m_q(\chi) = 2$ for some odd prime q dividing $|A|$. Then Lemma 6.5 yields that S centralizes every Sylow subgroup of A except one. Also Corollary 2.6 yields that χ_S is a sum of linear characters. As μ was an arbitrary extension of λ to $CA = C \times A$ with $\mu(z) = -1$, this means that $\nu^t = \nu$ for all $\nu \in \text{Lin}(C)$ with $\nu(z) = -1$. Thus S is abelian and G is as in Lemma 6.6 with $p = 2$ in this case.

Now assume that $\mathbf{C}_A(t) = 1$. If S is abelian and C is not just an elementary abelian 2-group, then again we find μ and χ as above, with $m_q(\chi) = 2$ for some odd prime q , and the result follows again.

If S is abelian and C is elementary 2-abelian, then $G = S[A, t]$ is generalized dicyclic.

Finally, assume that $\mathbf{C}_A(t) = 1$ and that S is nonabelian. Then S is generalized dicyclic, and S has an abelian subgroup D of index 2, such that $d^s = d^{-1}$ for all $d \in D$ and $s \in S \setminus D$. If $D = C$, then G is generalized dicyclic. Thus we may assume that $D \neq C$. If S is Dedekind, then $S \cong Q_8 \times (C_2)^r$, and we could choose $D = C$. So we can assume that S is not Dedekind, and thus $\mathbf{R}(S) = \langle z \rangle$ by Lemma 5.4. We may choose $t \in D \setminus C$ and $s \in C \setminus D$. Since both C and D are abelian, it follows that s centralizes $C \cap D$, and at the same time inverts the elements in $C \cap D$. Thus $C \cap D$ has exponent 2. Since $|S : C| = |S : D| = 2$ and $z \in \langle s \rangle \cap \langle t \rangle$, it follows $s^2 = t^2 = z$. Since $st \notin D$, we also have $(st)^2 = z$. It follows that $\langle s, t \rangle \cong Q_8$ and $S = \langle s, t \rangle \times (C \cap D) \cong Q_8 \times (C_2)^r$. But then S is Dedekind and G generalized dicyclic, contradiction. \square

The following is part of [3, Theorem 2(a)].

6.8. Lemma. *Let $G = PA$ be a Blackburn group such that $\mathbf{R}(G) \leq P \in \text{Syl}_p(G)$ and such that P and the normal p -complement A are abelian. Then we can write $P = \langle g \rangle \times P_0$, such that $\mathbf{C}_P(A) = \langle g^{p^c} \rangle \times P_0$ ($c \geq 1$), and $p^d := \text{ord}(g^{p^c})$ is the exponent of $\mathbf{C}_P(A)$. There is a $k \in \mathbb{N}$ such that $a^g = a^k$ for all $a \in A$.*

6.9. Lemma. *Let $G = PA$ be a Blackburn group, with a normal abelian p -complement A and $P \in \text{Syl}_p(G)$, where p divides $\mathbf{R}(G)$. Let q be a prime divisor of $|A|$, and H a q -complement in G . If $\chi \in \text{Irr}(G)$, then $m_q(\chi) = |\mathbb{Q}_q(\chi, \vartheta) : \mathbb{Q}_q(\chi)|$ for any irreducible constituent $\vartheta \in \text{Irr}(H)$ of χ_H .*

Proof. Let $Q \in \text{Syl}_q(G)$. Notice that $Q \trianglelefteq G$ and thus Q has a complement H in G . Let $\lambda \in \text{Lin}(Q)$ be a constituent of χ_Q . Then $K = \text{Ker}(\lambda)$ is normal in G (by definition of $\mathbf{R}(G)$) and thus $K \subseteq \text{Ker}(\chi)$. We may factor out K and assume without loss of generality that $K = 1$.

This means that Q is cyclic and thus χ is in a q -block with cyclic defect group. Thus we can apply Benard's theorem [2] to χ and conclude that $m_q(\chi) = |\mathbb{Q}_q(\chi, \varphi) : \mathbb{Q}_q(\chi)|$ for any irreducible Brauer constituent φ of χ . But an irreducible Brauer character of G contains the normal q -subgroup Q in its kernel, and thus can be identified with an ordinary character of the q' -group $H \cong G/Q$. Thus if φ is an irreducible Brauer constituent of χ , then $\varphi_H = \vartheta \in \text{Irr}(H)$ is an irreducible constituent of χ , and the result follows from Benard's theorem. \square

6.10. Lemma. *Let $G = PA$ be a Blackburn group, with a normal abelian p -complement A and $P \in \text{Syl}_p(G)$, where p divides $\mathbf{R}(G)$. Assume that $A = Q \times B$, where $B = \mathbf{C}_A(P)$ and $Q \in \text{Syl}_q(G)$, and set $C = \mathbf{C}_P(Q)$. Any nonlinear $\chi \in \text{Irr}(G)$ has the form $\chi = (\mu \times \lambda)^G$ for some $\mu \in \text{Lin}(CB)$ and $\lambda \in \text{Lin}(Q)$. Let $\vartheta \in \text{Lin}(PB \mid \mu)$. Then $m_q(\chi) = \ell/k$, where ℓ is the smallest positive integer such*

that $\text{ord}(\vartheta)$ divides $q^\ell - 1$, and k is the smallest positive integer such that $\text{ord}(\mu)$ divides $q^k - 1$.

(In other words, ℓ and k are the multiplicative orders of q modulo $\text{ord}(\vartheta)$ and modulo $\text{ord}(\mu)$, respectively.)

Proof of Lemma 6.10. Notice that $CB = \mathbf{Z}(G)$, and that $H = PB$ is an abelian q -complement. Let $\chi \in \text{Irr}(G)$. If $Q \leq \text{Ker}(\chi)$, then χ is linear, since $G/Q = P \times B$ is abelian. (In fact, $Q = G'$.) Otherwise, let $\lambda \neq 1$ be a linear constituent of χ_Q . Then $G_\lambda = CBQ = CA$, and χ is induced from some linear character of the abelian group CBQ , say $\chi = (\mu \times \lambda)^G$ with $\mu \in \text{Lin}(CB)$. It follows that $\mathbb{Q}_q(\chi) = \mathbb{K}(\mu)$, where $\mathbb{K} \subseteq \mathbb{Q}_q(\lambda)$ is totally ramified over \mathbb{Q}_q , and the extension $\mathbb{K}(\mu)/\mathbb{K}$ is unramified. By the general form of unramified extensions, the residue field of $\mathbb{Q}_q(\chi) = \mathbb{K}(\mu)$ has order q^k , where k is the smallest positive integer such that $\text{ord}(\mu)$ divides $q^k - 1$.

The restriction χ_H to the q -complement $H = PB$ is the sum of all linear characters $\vartheta \in \text{Lin}(H)$ lying over μ . Thus $\mathbb{Q}_q(\chi, \vartheta) = \mathbb{K}(\vartheta)$ is generated by $\mathbb{Q}_q(\chi)$ and a root of unity of order $\text{ord}(\vartheta)$. Since $\text{ord}(\vartheta)$ is not divisible by q , the extensions $\mathbb{Q}_q(\chi, \vartheta)/\mathbb{Q}_q(\chi)$ and $\mathbb{K}(\vartheta)/\mathbb{K}$ are unramified. We can thus compute $|\mathbb{Q}_q(\chi, \vartheta) : \mathbb{Q}_q(\chi)|$ by computing the degrees of the residue fields. As above, the residue field of $\mathbb{Q}_q(\chi, \vartheta) = \mathbb{K}(\vartheta)$ has order q^ℓ , where ℓ is the smallest positive integer such that $\text{ord} \vartheta$ divides $q^\ell - 1$. Now the result follows from Lemma 6.9. \square

6.11. Lemma. *Let $G = (PQ) \times B$ be as in Theorem D (iv). Then $\text{NKer}_{\mathbb{Q}_q}(G) \neq 1$. (More precisely, $\text{NKer}_{\mathbb{Q}_q}(G) \cap \langle g \rangle \neq 1$, with $g \in P$ as in Lemma C.)*

Notice that this contains Lemma C from the introduction.

Proof. Recall that $P = \langle g \rangle \times P_0$, where g has order p^{c+d} and $C := \mathbf{C}_P(Q) = \langle g^{p^c} \rangle \times P_0$. Moreover, we assume that $(q-1)_p$ divides p^d . Let $z \in \langle g^{p^c} \rangle$ be an element of order p . We claim that $z \in \text{NKer}_{\mathbb{Q}_q}(G)$.

Suppose that $\chi(z) \neq \chi(1)$ for $\chi \in \text{Irr } G$. If $\chi(1) > 1$, then $\chi = (\mu \times \lambda)^G$ as in Lemma 6.10, with $\mu \in \text{Lin}(CB)$ and $1 \neq \lambda \in \text{Lin } Q$.

To compute the Schur index of such a χ , we apply Lemma 6.10. Since $\mu(z) \neq 1$, it follows that $\text{ord}(\mu) = p^d n$, where n divides the exponent of B . For $\vartheta \in \text{Lin}(PB \mid \mu)$, we have $\text{ord}(\vartheta) = p^c \text{ord}(\mu) = p^{c+d} n$. Let f be the order of q modulo p^d , so that p^d divides $q^f - 1$. As $(q-1)_p$ divides p^d , it follows that $(q^f - 1)_p = p^d$. Thus the order of q modulo p^{c+d} is $p^c f$.

Let k and l be the multiplicative order of q modulo $p^d n$ and $p^{c+d} n$, respectively. The assumption in (iv) in Theorem D yields that k/f is not divisible by p . Thus we must have $l/k \geq p^c$ and thus $m_q(\chi) = p^c = \chi(1)$. This was to be shown. \square

Proof of Theorem D. By Theorem 6.2, Theorem 6.3 and Lemma 6.11, each of the conditions in Theorem D ensures that $\text{NKer}(G) \neq 1$.

Conversely, assume that $\text{NKer}(G) \neq 1$. By Lemma 6.1, we have that $G = PA$, where $P \in \text{Syl}_p(G)$ and A is a normal p -complement and a Dedekind group. By Theorem 6.2, we may assume that A is abelian. By Theorem 6.3, we can assume that G is not nilpotent, and thus $\mathbf{C}_P(A) < P$. By Lemmas 6.6 and 6.7, we can assume that $G = (PQ) \times B$, and that P is also abelian. Thus Lemma 6.8 applies. Write $P = \langle g \rangle \times P_0$ and $\mathbf{C}_P(A) = \langle g^{p^c} \rangle \times P_0$, as in that lemma.

Let $z \in \mathbf{R}(G) \subseteq \langle g^{p^c} \rangle$ have order p and notice that we must have $z \in \text{NKer}(G)$, since $1 \neq \text{NKer}(G) \leq \mathbf{R}(G)$. Thus we must have $m_{\mathbb{Q}}(\chi) = \chi(1)$ for any $\chi \in \text{Irr}(G)$ with $\chi(z) \neq \chi(1)$. Suppose $\chi = (\mu \times \lambda)^G$ as in Lemma 6.10. By the local-global principle (Lemma 2.5 (i)), the Schur index of χ over some local field must equal $\chi(1) = |P : C| = p^c$. We claim that this local field must be \mathbb{Q}_q at least for some nonlinear χ . If $p^c > 2$, then we must have $m_q(\chi) = \chi(1)$, by Lemma 6.5. In the case where $p^c = 2$, choose $\mu \in \text{Lin}(CB)$ of order greater than 2. This is possible since when the exponent of $C \times B = \mathbf{Z}(G)$ divides 2, then G is generalized dicyclic, and the proof is finished. We have then $m_{\mathbb{R}}(\chi) = 1$. Also, χ is induced from a subgroup of index 2 (which is not a 2-group), and thus χ remains irreducible after reducing mod 2. Thus $m_2(\chi) = 1$ by Lemma 2.5 (iii). Thus in every case, we must have $m_q(\chi) = \chi(1)$.

We can now apply Lemma 6.10. Notice that since $\mu(z) \neq 1$ by assumption, we have $\text{ord}(\vartheta) = p^c \text{ord}(\mu)$, as can be seen from the structure of P (Lemma 6.8), and $\text{ord}(\mu) = p^d \cdot n$, where n divides the exponent of B , and thus is not divisible by p . We may choose μ such that n equals the exponent of B . Let f be the order of $q \bmod p^d$ and let k' be the order of $q^f \bmod n$. Then the order of $q \bmod p^d n$ is $k = fk'$. If $p^d < (q - 1)_p$, or if p divides k' , then certainly $p^d < (q^k - 1)_p$. But this yields that $q^{kp^{c-1}} \equiv 1 \bmod p^{c+d}n$, and thus Lemma 6.10 yields that $m_q(\chi) < p^c$, contradiction. This means that (iv) in Theorem D holds. \square

ACKNOWLEDGMENT

I wish to thank Erik Friese for a thorough reading of this paper and many useful remarks.

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